Tunnelling

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In https://twitter.com/DavidDeutschOxf/status/523061654326894592, you wrote: "'Quantum tunnelling' is very badly named. It's mountaineering: In some universes the system has enough energy to cross the barrier. Period."

This claim is wrong. In a typical tunnelling experiment there is no region in which the energy of the tunnelling particle is larger than the energy of the barrier. The explanation of tunnelling has nothing to do with any instance of the particle anywhere in the multiverse having energy greater than that of the barrier. Rather, tunnelling is in part a consequence of the principle of locality. There is no way for the particle to get information about the potential in the barrier without some instances of it going into the barrier and carrying information (in the sense of dependence of the wave function on the potential in the barrier) about the inside of the barrier to the instances of the particle that are incident on the barrier.

To illustrate the problem, I will consider the case of a one-dimensional potential $V(x)$ that has constant value V_0 in the region $0 < x < L$, with no potential elsewhere. This treatment follows (Hartman,1962). The equation of motion for the wave function of the particle in such a system is given by

$$\frac{\hbar^2}{2m}\frac{\partial^2\Psi(x,t)}{\partial x^2} - V(x)\Psi(x,t) = -i\hbar\frac{\partial\Psi(x,t)}{\partial t}. \tag{1}$$

The solutions to this equation can be found by constructing the stationary state solutions using the continuity of the wave function and its first derivative at $x = 0$ and $x = L$, as in standard undergraduate textbook quantum mechanics. These solutions are superposed to form a time dependent state by multiplying them by

$\exp(iEt/\hbar)$ multiplying them by a weighting function, applying an initial position condition and integrating over all the possible states.

The energy of a wave packet with wave number k is given by

$$E(k) = \frac{\hbar^2 k^2}{2m}. \tag{2}$$

The weighting function is

$$f(k) = \exp\left(-\frac{(k-k_0)^2}{\Delta^2}\right) \tag{3}$$

So if $k_0 << k_v, \Delta << k_v$ then $f(k)$ is very small for values of the energy above V_0.

Define

$$\begin{aligned} k_v &= \frac{\sqrt{2mV_0}}{\hbar} \\ \kappa &= \frac{\sqrt{2m(V_0 - E(k))}}{\hbar}. \end{aligned} \tag{4}$$

The solution to the wave packet's equation of motion is given by:

$$\begin{aligned} \Psi(x,t) &= \eta_1(x,t) + \chi_1(x,t) \ (x \le 0) \\ \Psi(x,t) &= \eta_2(x,t) + \chi_2(x,t) \ (0 \le x \le L) \\ \Psi(x,t) &= \eta_3(x,t) \ (L \le x), \end{aligned} \tag{5}$$

In $x \le 0$ we have

$$\eta_1(x,t) = \frac{1}{\Delta\sqrt{2\pi}} \int_0^\infty f(k) \exp i\left(k(x+x_0) - \frac{E(k)t}{\hbar}\right) dk \tag{6}$$

$$\begin{aligned} \chi_1(x,t) = \frac{1}{\Delta\sqrt{2\pi}} \int_0^{k_v} f(k) B_1(k) \exp i\left(k(-x+x_0) + \beta_1(k) - \frac{E(k)t}{\hbar}\right) dk \\ + \frac{1}{\Delta\sqrt{2\pi}} \int_{k_v}^\infty f(k) D_1(k) \exp i\left(k(-x+x_0) + \delta_1(k) - \frac{E(k)t}{\hbar}\right) dk. \end{aligned}$$

In the second region $0 \le x \le L$:

$$\eta_2(x,t) = \frac{1}{\Delta\sqrt{2\pi}} \int_0^{k_v} f(k) A_2(k) \exp\left(-\kappa x + ikx_0 + i\alpha_2(k) - i\frac{E(k)t}{\hbar}\right) dk$$
$$+\frac{1}{\Delta\sqrt{2\pi}} \int_{k_v}^{\infty} f(k) C_2(k) \exp\left(-\kappa x + ikx_0 + i\gamma_2(k) - i\frac{E(k)t}{\hbar}\right) dk \quad (7)$$
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$$+\frac{1}{\Delta\sqrt{2\pi}} \int_{k_v}^{\infty} f(k) D_2(k) \exp\left(\kappa x + ikx_0 + i\delta_2(k) - i\frac{E(k)t}{\hbar}\right) dk.$$

In the third region $L \le x$:

$$\eta_3(x,t) = \frac{1}{\Delta\sqrt{2\pi}} \int_0^{k_v} f(k) A_3(k) \exp i\left(k(x+x_0) + \alpha_3(k) - \frac{E(k)t}{\hbar}\right) dk$$
$$+\frac{1}{\Delta\sqrt{2\pi}} \int_{k_v}^{\infty} f(k) C_3(k) \exp i\left(k(x+x_0) + \gamma_2(k) - \frac{E(k)t}{\hbar}\right) dk. \quad (8)$$

The coefficients are given by

$$\begin{aligned} A_3(k) &= 2k\kappa F(k) \\ B_1(k) &= (k^2+\kappa^2)\sinh(L\kappa) F(k) \\ C_3(k) &= 2ik\kappa G(k) \\ D_1(k) &= -i(k^2+\kappa^2)\sinh(L\kappa) G(k) \\ F(k) = 1/&\sqrt{4k^2\kappa^2\cosh^2(L\kappa) + (k^2-\kappa^2)^2\sinh^2(L\kappa)} \\ G(k) = 1/&\sqrt{4k^2\kappa^2\cosh^2(L\kappa) + (\kappa^2-k^2)\sinh^2(L\kappa)}. \end{aligned} \quad (9)$$

The phases are given by

$$\begin{aligned} \alpha_3(k) &= \tan^{-1}\left(\frac{k^2-\kappa^2}{2k\kappa}\tanh L\kappa\right) - Lk \\ \beta_1(k) &= \tan^{-1}\left(\frac{2k\kappa}{\kappa^2-k^2}\coth(L\kappa)\right) \\ \gamma_3(k) &= \tan^{-1}\left(\frac{k^2+\kappa^2}{2k\kappa}\tan L\kappa\right) - Lk \\ \delta_1(k) &= \tan^{-1}\left(\frac{-2k\kappa}{\kappa^2+k^2}\cot(L\kappa)\right). \end{aligned} \quad (10)$$

The wave packet starts far from the barrier in the sense that its tail near the barrier is small. Let's consider a wave packet in empty space with no potential $\Theta(x,t)$ with the same momentum distribution centred at a distance x_0 from the origin. We have

$$\Theta(x,t) = \frac{1}{\Delta\sqrt{2\pi}} \int_{-\infty}^{\infty} f(k) \exp i \left(k(x+x_0) - \frac{E(k)t}{\hbar} \right) dk, \tag{11}$$

which gives

$$\Theta(x,t) = \sqrt{\frac{\pi}{i(\hbar/2m)t + (1/\Delta^2)}} \exp(i(k_0 x - \omega_0 t)) \exp\left(-\frac{(x - x_0 - v_g t)^2}{4(i(\hbar/2m)t + (1/\Delta^2))} \right). \tag{12}$$

The probability density for the wave packet is

$$|\Theta(x,t)|^2 = \frac{\pi}{(\hbar/2m)^2 t^2 + (1/\Delta^2)^2} \exp\left(-\frac{(1/\Delta^2)(x - x_0 - v_g t)^2}{2(\hbar/2m)^2 t^2 + (1/\Delta^2)^2} \right). \tag{13}$$

This is an approximation to the state $\eta_1(x,t)$, which is the forward propagating portion of the wave packet for $x < 0$. To get the wave packet's spread in momentum after time t we would do the inverse of the above procedure at that time and so we can expect the spread in momentum at that time to be something like $\sqrt{2(\hbar/2m)^2 t^2 + (1/\Delta^2)^2}$. So the increase in the spread is approximately $(\hbar/2m)^2\tau$ where τ is the time taken to reach the origin. If we want the wave packet to have negligible amplitude near the origin at $t = 0$ then we need $x_0 = n/\Delta^2$ for some integer n which we can assume is less than 10. The wave packet takes a time $\tau = x_0/v_g$ to reach the origin so the spread in frequency by the time it reaches the barrier is $\beta\tau = n\Delta^2/2k_0$.

Does this wave packet act as we would expect? That is, is it negligible for $x > 0$ at $t = 0$ and does it arrive at a reaonable time? The $\chi_1(x,t)$ term will be similar with x swapped for $-x$ so that it will represent a backward propagating wave packet. The Gaussian term in $\chi_1(x,t)$ will not be one until $x = x_0 + v_g t$ unlike the forward propagating $\eta_1(x,t)$ which is one when $x = x_0 - v_g t$. So the time delay between the peaks of the forward and backward progating wave packets is $2v_g\tau$ as we would expect. The wave packet on the far side of the barrier will also be of a similar form to $\Theta(x,t)$ with a relatively slowly varying k-dependent amplitude, as I discuss below, and so will not arrive until $x = x_0 - v_g t$ for an $x > L$, i.e. - until after $v_g t > L - x_0$ and since x_0 is negative this means $v_g t > L$, which is what we would expect. In particular it is small at $t = 0$.

What the wave function about inside the barrier at $t = 0$? Inside the potential, the integral is over something like $f(k)$ times $e^{\kappa x}$ and $e^{-\kappa x}$ times slowly wavering

k-dependent terms as discussed below. Now, κ is approximately linear in k, and x is at most a small multiple of kappa. So the integral will change less with respect to x than would the Θ state. In addition the largest value of x in the potential is some small multiple of κ so neither the of the $+\kappa, -\kappa$ states will change much in amplitude. The amplitude at the boundary $x = 0$ is the same for the $x < 0$ and $x > 0$ states by virtue of the boundary conditions at $x = 0$ for the energy eigenstates, so it is very small at the boundary. As a result it is small throughout the block.

I will take this as an approximation to what would happen for a wave packet propagating from x_0 before the origin to the barrier. Since Δ and k_0 are of similar order of magnitude $\beta\tau \approx nk_0/2$. So for the wave packet above the spread in frequencies owing to the wave packet propagating to the barrier is negligible. So the amplitude for frequencies above the barrier will still be low.

You might be concerned that the amplitude terms in the above expressions might produce a significant amplitude above the barrier. I will not evaluate them fully, but I will argue that their amplitude for energies above the barrier potential is negligible. Before and after the barrier the amplitude is modified by the amplitude expressions given above. In the barrier those expressions are modified by both the amplitudes and terms of the form $e^{\kappa x}, e^{-\kappa x}$.

First let's deal with the $e^{\kappa x}, e^{-\kappa x}$ terms. Above V_0, κ is imaginary and so this is a phase term and does not increase the magnitude of energies above the barrier. Below the barrier, κ is real and the part that varies with k is roughly similar in magnitude to k. The barrier must be of the order $1/\kappa$ in thickness for any tunnelling to take place so κx has to be something like κ or $1/\kappa$. So both terms are not significant compared to $f(k)$.

The amplitude terms are all going to be dominated by terms of the form $\sinh(\kappa L), \cosh(\kappa L)$ since the other terms are of order k^2. So the amplitude terms are dominated by expressions of the form $e^{\kappa L}, e^{-\kappa L}$, which are not significant compared to $f(k)$.

So the amplitude for energies above the barrier is negligible. Instances of the particle get through to the other side of the barrier despite having lower energy than the barrier.

References

Hartman, T. E., 1962, Tunneling of a wave packet, Journal of Applied Physics, Vol. 33, pp. 3427–3433.